2-D Vectors, Kinematics

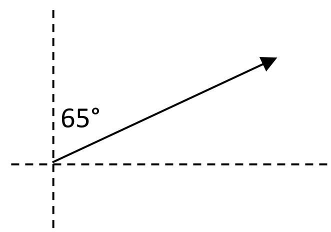
Topics covered:

- Vectors:
  - Adding/Subtracting vectors (vector diagrams)
  - 2-D perpendicular vectors
- Adding vectors component method
- Adding Vectors Cosine & Sine Laws Method
- Projectile motion (1-D & 2-D)
- Frames of Reference (inertial and non-inertial)
- Relative motion

Vectors

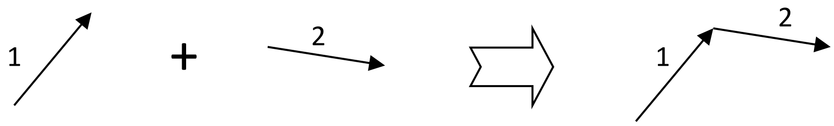
Vectors are quantities that have both magnitude and direction.
They can be represented graphically as arrows:

Length of arrow: magnitude of vector
Direction of arrow: direction of vector e.g. 15 km [N65°E]

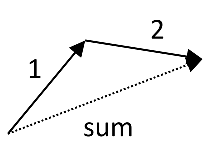


Adding vectors

To add two vectors, place the tail of the second to the head of the first.



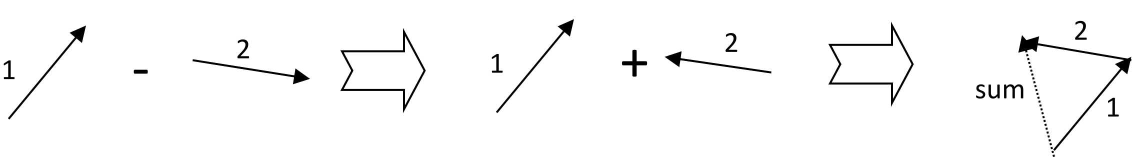
The sum is the vector drawn from the tail of the first (start) to the head of the second (end)



Subtracting vectors

To subtract vectors, consider it as adding a negative vector. A negative vector means its direction has been reversed:

$$\vec{v}_1 - \vec{v}_2 = \vec{v}_1 + (-\vec{v}_2)$$



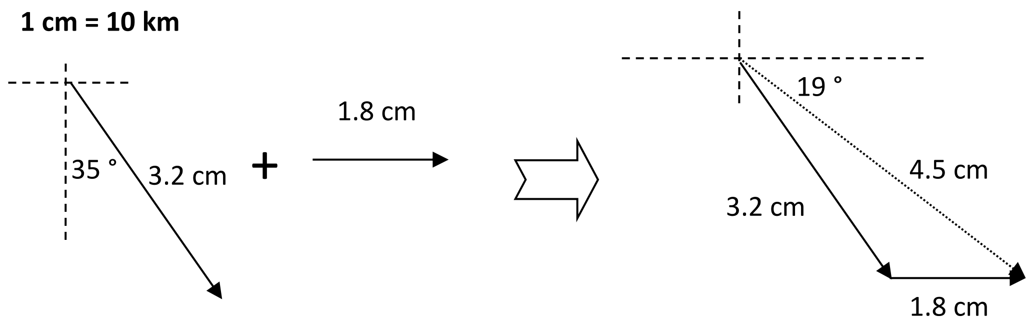
Vector (Scale) Diagrams

Vectors can be added (or subtracted) visually by drawing scale diagrams in the following way:

1. An appropriate scale must be chosen such that the vectors will fit on the page
2. The vectors are drawn (tail to head) to this scale, with their directions preserved
3. The resultant vector (sum) is drawn (tail to head)
4. The resultant vector 's length is measured and converted back into original units to determine its magnitude and its angle measured to determine its direction:

e.g. A bird flies 32 km [S 35° E] and then 18 km [E]. Find the total displacement of the bird.

Scale: 1 cm = 10 km



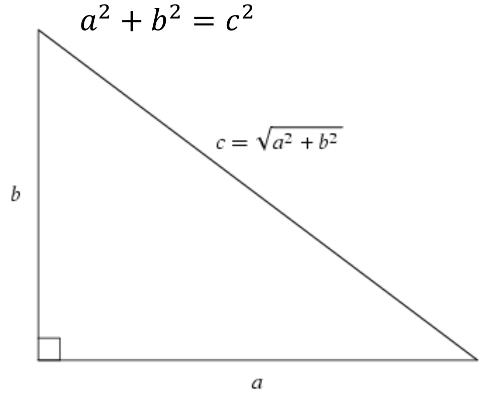
$\Delta\vec{d} = 4.5\ km\ [E\ 19\,°\ E]$

2-D perpendicular vectors

To add vectors at right angles to each other algebraically, you will need to use:

Pythagoras' Theorem

$$a^2 + b^2 = c^2$$



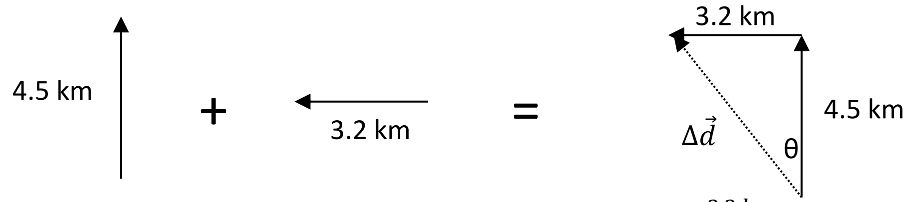
SOH CAH TOA

$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$

$$\tan\theta = \frac{opposite}{adjacent}$$

e.g. A human moves 4.5 km [N] and then 3.2 km [W]. Find the total displacement of the human



$$\Delta d = [(4.5\ km)^2 + (3.2\ km)^2]^{½}$$

$$\Delta d = 5.5\ km$$

$$\Delta\vec{d} = 5.5\ km\ [N\ 35\,°\ W]$$

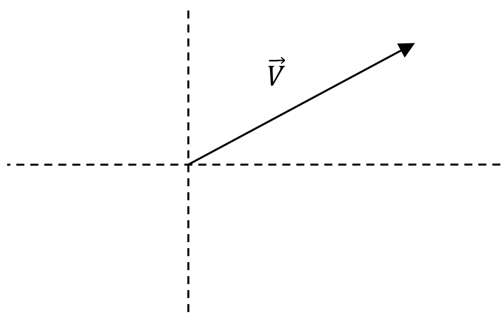
$$\tan\theta = \frac{3.2\ km}{4.5\ km}$$

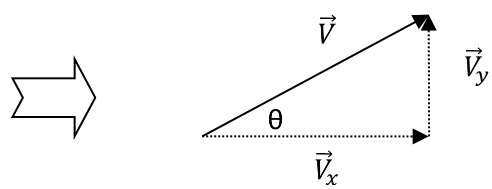
$$\theta = \tan^{-1}\left(\frac{3.2\ km}{4.5\ km}\right)$$

$$\theta = 35\,°$$

Adding 2-D Vectors – Component Method

Any vector can be broken into its *x* and *y* components:

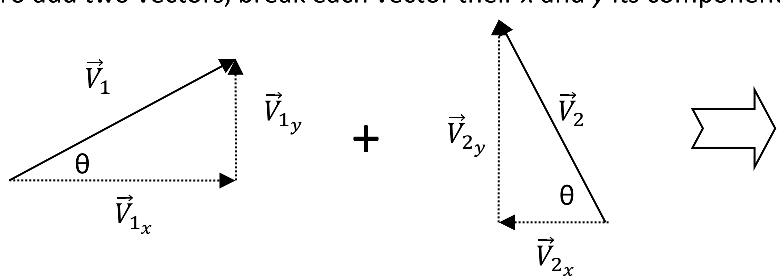




where $\vec{V}_x = \vec{V}\cos\theta$

$\vec{V}_y = \vec{V}\sin\theta$

To add two vectors, break each vector their *x* and *y* its components and add the magnitudes separately:



$$\vec{V}_{T_x} = \vec{V}_{1_x} + \vec{V}_{2_x}$$

$$\vec{V}_{T_x} = \vec{V}_1\cos\theta + \vec{V}_2\cos\theta$$

$$\vec{V}_{T_y} = \vec{V}_{1_y} + \vec{V}_{2_y}$$

$$\vec{V}_{T_y} = \vec{V}_1\sin\theta + \vec{V}_2\sin\theta$$

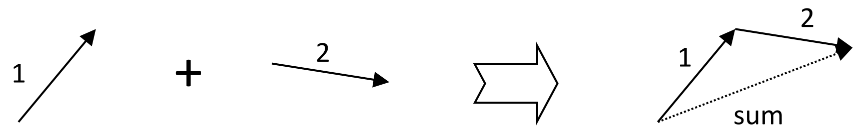
*notice that as $\vec{V}_{2_x}$ is in the opposite direction as $\vec{V}_{1_x}$, it will be negative.

Now the x and y components of the vector sum ($\vec{V}_T$) have been determined. To find the actual vector sum, it is just like adding 2-D perpendicular vectors: Pythagoras' theorem and SOHCAHTOA are used:

$$\vec{V}_T = \left[\left(\vec{V}_{T_x}\right)^2 + \left(\vec{V}_{T_y}\right)^2\right]^{½} \qquad \theta = \tan^{-1}\left(\frac{\vec{V}_{T_y}}{\vec{V}_{T_x}}\right)$$

Adding 2-D Vectors - Cosine & Sine Laws Method

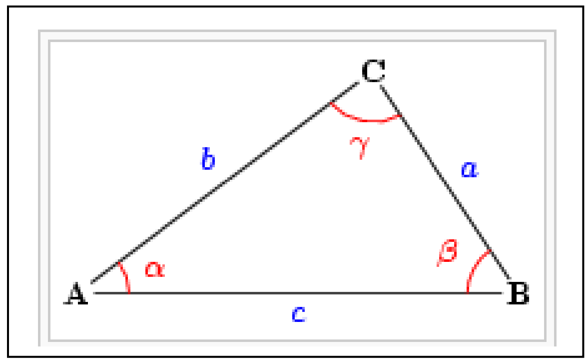
When adding two (and only two) vectors, a triangle is created with the third side being the vector sum:



The law of Cosines and Sines can now be used to solve for any unknown.

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma),$$
$$b^2 = a^2 + c^2 - 2ac\cos(\beta),$$
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$



Law of Sines

$$\frac{\sin\alpha}{A} = \frac{\sin\beta}{B} = \frac{\sin\gamma}{C}$$

1-D Projectile Motion (Objects in Free Fall)

A projectile is any object whose change in motion (acceleration) is caused only by the gravitational pull by a very large object (like the earth!). A thrown pie, once it has left your hand, is a perfect example of a projectile (except for the wasting of a delicious food part).

Galileo Galilei, over 350 years ago, was really interested in objects in freefall, and found out two very important things (if all friction, including air resistance, is ignored):

1. The rate at which ANY object accelerates due to gravity is INDEPENDENT of its mass. (Mass does not appear in any of the 5kinematic equations for uniform linear acceleration above)

2. The rate at which ANY object accelerates during free fall (near the surface of the earth) is ALWAYS $\vec{g}$ (little 'g'), where $\vec{g} = 9.8\,m/s^2\ [towards\ the\ centre\ of\ the\ earth/down]$

Facts to remember:

- Acceleration due to gravity is constant throughout the object's motion in magnitude and direction ($\vec{g}$ is always* 9.8 $^{m}/_{s^2}$ [down] 'near' the surface of the earth)

- At the maximum height of the object's motion, its velocity = 0 m/s (and its acceleration remains $\vec{g} = 9.8\ m/s^2\ [down]$)

- Because an object's acceleration is constant, Δt to reach max height = Δt to return from max height

Equations for Uniformly Accelerated Motion:

$$\vec{v}_2 = \vec{v}_1 + \vec{a}\Delta t \qquad \text{or} \qquad \vec{a} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t}$$

$$\Delta\vec{d} = \tfrac{1}{2}(\vec{v}_1 + \vec{v}_2)\Delta t \qquad \text{or} \qquad \Delta\vec{d} = \vec{v}_{avg}\Delta t$$

$$\Delta\vec{d} = \vec{v}_1\Delta t + \tfrac{1}{2}\vec{a}\Delta t^2$$

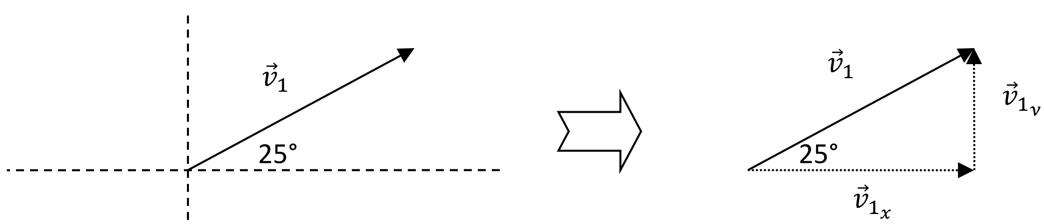
$$\Delta\vec{d} = \vec{v}_2\Delta t - \tfrac{1}{2}\vec{a}\Delta t^2$$

$$v_2^2 = v_1^2 + 2a\Delta d$$

<u>**2-D Projectile Motion (Objects in Free Fall)**</u>

To solve 2-D projectile motion problems, classify all motion variables as either occurring in the vertical (*y*) direction or the horizontal (*x*) direction. Those variables that occur in ***both*** directions must be broken up into their *x* and *y* components:

e.g. $\vec{v}_1$ = 3.2 m/s [25°up from horizontal]



$$\vec{v}_{1_x} = \vec{v}_1 cos\theta \qquad \vec{v}_{1_y} = \vec{v}_1 sin\theta$$
$$\vec{v}_{1_x} = (3.2\ m/s)\cos 25° \qquad \vec{v}_{1_y} = (3.2\ m/s)\sin 25°$$
$$\vec{v}_{1_x} = 2.9\ m/s \qquad \vec{v}_{1_y} = 1.4\ m/s$$

Facts to remember:

- All motion in the vertical direction has no influence on the motion in the horizontal direction (and vice versa), so the two directions can be analyzed independently

- Gravitation is the only force that acts on a projectile, and so will always experience a constant acceleration downwards

- As there are no forces acting the horizontal direction (ignoring frictional forces), the motion in the *x*-direction is uniform ($\vec{a} = 0$)

- The time it takes for the motion to occur in the *x*-direction is the same as that in the *y*-direction

Once all of the projectile's motion has been separated into horizontal and vertical directions, the equations for uniformly accelerated motion can be used to determine what is being asked for.

e.g. A ball is thrown out of a building from a 2.5 m high window with an initial velocity of 3.2 m/s [25°up from horizontal]. How far from the base of the building will the ball hit the ground?

x-direction	*y*-direction
Δt = ?	Δt = ?
	$\vec{a} = 9.8\frac{m}{s^2}[down]$
$\Delta\vec{d}_x = ?$	$\Delta\vec{d}_y = 2.5\ m\ [down]$
$\vec{v}_x = \vec{v}_1 \cos\theta$	$\vec{v}_{1_y} = \vec{v}_1 \sin\theta$
$= (3.2\ m/s)\cos 25°$	$= (3.2\ m/s)\sin 25°$
$= 2.9\ m/s\ [\rightarrow]$	$= 1.4\ m/s\ [up]$

From the *y*-direction:

$$\Delta\vec{d} = \vec{v}_1\Delta t + \tfrac{1}{2}\vec{a}\Delta t^2$$

$$\tfrac{1}{2}\vec{a}\Delta t^2 + \vec{v}_1\Delta t - \Delta\vec{d} = 0$$ quadratic equation of variable Δt

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = \tfrac{1}{2}\vec{a}$, $b = \vec{v}_1$, $c = -\Delta\vec{d}$

$a = -4.9\frac{m}{s^2}$, $b = 1.4\ m/s$, $c = 2.5\ m$

From the quadratic formula: Δt = -0.59 s, Δt = 0.87 s

Discarding the negative time interval value and using the information from the *x*-direction, we can now determine the *range* ($\Delta\vec{d}_x$) of the flight of the ball:

$$\vec{v} = \frac{\Delta\vec{d}}{\Delta t}$$

$$\Delta\vec{d}_x = \vec{v}_x\Delta t$$

$$\Delta\vec{d}_x = \left(\frac{2.9m}{s}[\rightarrow]\right)(0.87\ s)$$

$$\Delta\vec{d}_x = 2.5\ m\ [\rightarrow]$$

Frames of Reference

All motion is relative, which means, the motion of an object must be compared to something in order to be meaningful. In most cases, when we describe the motion of an object, it is in relation to the earth.

'The car was travelling 80 km/h' ⇒ 'The car was travelling 80 km/h *relative to the earth'*

Although the first statement is most likely to be heard, the second statement is more correct, as the speed of the car according to another car would be quite different:

e.g. A car is moving at 80 km/h [N] and a truck is moving at 90 km/h [N]. How fast is the car moving:

a) relative to the earth? ANS: 80 km/h [N]
b) relative to the truck? ANS: -10 km/h [N] or 10 km/h [S]

In the example above, the earth is in one *frame of reference* and the truck is in another *frame of reference*. Properly defined, a **frame of reference is a coordinate system relative to which motion can be observed**.

If the frame of reference and the motion observed are not accelerating with respect to each other, the frame of reference is an ***inertial frame of reference*** (e.g. earth can car moving at constant speed)

Even though observers in different inertial frames of references will disagree about the values of motion (as above), the laws of physics used to find the values are the same.

If the frame of reference and the motion observed are accelerating with respect to each other, the frame of reference is a ***non-inertial frame of reference*** (e.g. earth and car turning a corner)

Observers in different non-inertial frames of reference will disagree about the values but also the laws of motion used to find them (Newton's first law will be violated – to account for this, 'fictitious forces' must be created to 'correct' Newton's first law).

Relative Motion

To solve relative motion problems (involving **inertial frames of reference** only), adding vectors is required, according to the following formula:

$$\vec{v}_{AE} = \vec{v}_{AB} + \vec{v}_{BE}$$

Which reads:

the velocity of object A (*A*) relative to the earth (*E*) is equal to the velocity of object A (*A*) relative to object B (*B*) plus the velocity of object B (*B*) relative to the earth (*E*)

This equations works for all relative motion problems, though the subscript letters (*A*,*B*, etc) are usually changed to represent what the objects themselves are.

e.g. (from above) A car is moving at 80 km/h [N] and a truck is moving at 90 km/h [N]. How fast is the car moving relative to the truck?

$$\vec{v}_{CE} = \vec{v}_{CT} + \vec{v}_{TE}$$

or

$$\vec{v}_{Car\ relative\ to\ Earth} = \vec{v}_{Car\ relative\ to\ Truck} + \vec{v}_{Truck\ relative\ to\ Earth}$$

We are looking for $\vec{v}_{CT}$ and already know $\vec{v}_{CE}$ and $\vec{v}_{TE}$, so:

$$\begin{aligned} \vec{v}_{CT} &= \vec{v}_{CE} - \vec{v}_{TE} \\ &= 80\ km/h\ [N] - 90\ km/h\ [N] \\ &= -10\ km/h\ [N] \end{aligned}$$

In some cases, object B is not an object at all, but a moving medium through which an object travels (and influences its motion) (e.g., boat across water, plane through the air, etc). The relative motion is the same, but the subscripts will change:

$$\vec{v}_{OE} = \vec{v}_{OM} + \vec{v}_{ME}$$

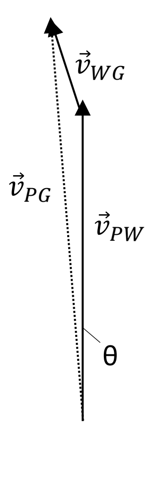
or

$$\vec{v}_{Object\ relative\ to\ Earth} = \vec{v}_{Object\ relative\ to\ Medium} + \vec{v}_{Medium\ relative\ to\ Earth}$$

e.g. A plane takes off flying north with a speed of 85.0 m/s into a wind that travels at 25.0 m/s [N 18° W]. Determine the velocity of the plane according to an observer on the ground.
(P = plane, W = wind, G = ground)

$$\vec{v}_{PW} = 85.0\ m/s\ [N]$$
$$\vec{v}_{WG} = 25.0\ m/s\ [N\ 18°W]$$
$$\vec{v}_{PG} = ?$$

$$\vec{v}_{PG} = \vec{v}_{PW} + \vec{v}_{WG}$$



$$\begin{aligned} v_{PG} &= [(v_{PW})^2 + (v_{WG})^2 - 2(v_{PW})(v_{WG})\cos 162°]^{½} \\ &= 109\ m/s \end{aligned}$$

$$\theta = \sin^{-1}\left[(v_{WG})\left(\frac{\sin 162°}{v_{PG}}\right)\right]$$
$$\theta = 4.06°$$

$$\vec{v}_{PG} = 109\ m/s\ [N\ 4.06°W]$$