Resolution

This short chapter deals in detail with the limits to resolution imposed by diffraction.

Objectives

By the end of this chapter you should be able to:

- understand what is meant by *resolution;*
- apply the *Rayleigh criterion*.

The Rayleigh criterion

In the previous chapter, we discussed in some detail the diffraction of a wave through a slit of linear size *b*. One application of diffraction is in the problem of the resolution of the images of two objects that are close to each other.

Light from a distant star will, upon passing through a lens, diffract around the circular aperture of the lens. The image of a star is an extended disc with diffraction rings around it. Two distant objects that are very close to each other will, in general, produce diffraction patterns that will merge with each other, making it difficult to distinguish the pattern as one belonging to two separate objects (see Figure 8.1).

Figure 8.1 Diffraction limits our ability to distinguish two separate sources. In the first diagram the diffraction patterns have merged.

Rayleigh suggested that a useful criterion for deciding whether the two objects can be resolved is that the central maximum of one of the sources is formed at the position of the first minimum in the diffraction pattern of the other (see Figure 8.2).

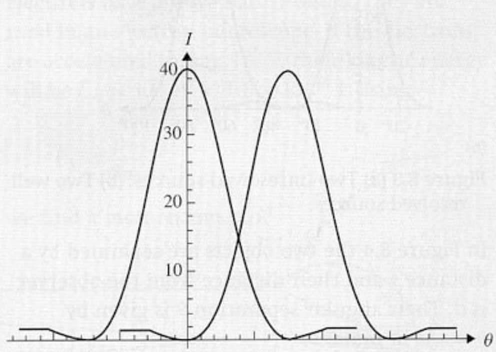

Figure 8.2 The Rayleigh criterion states that two sources are just resolved if the central maximum of the diffraction pattern of one source falls on the first minimum of the other.

Figure 8.3 shows two unresolved and two well-resolved sources.

▶ Recall that the first minimum in the diffraction pattern through a circular aperture of size b is formed at an angle θ given by

$$\theta = 1.22\frac{\lambda}{b}$$

and for a square aperture of side b by

$$\theta = \frac{\lambda}{b}$$

It then follows that the two objects can be resolved if their angular separation is larger than the θ given by the diffraction formulae above.

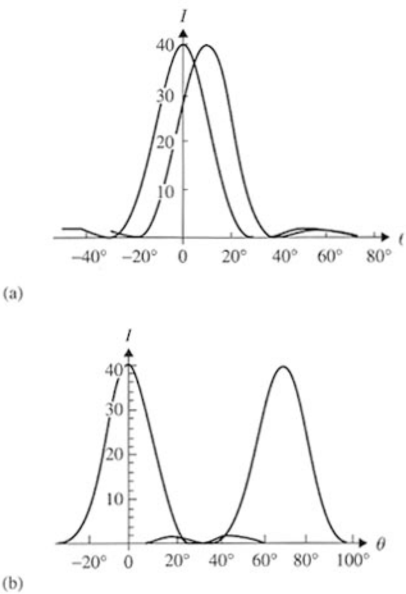

Figure 8.3 (a) Two unresolved sources. (b) Two well-resolved sources.

In Figure 8.4 the two objects are separated by a distance s and their distance from the observer is d. Their angular separation θ is given by $\theta = s/d$ in radians.

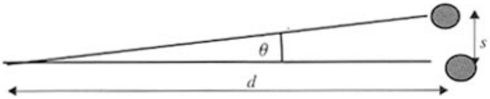

Figure 8.4 To see the two objects as distinct we need a lens that can resolve the angle θ.

Example questions

Q1

The camera of a spy satellite orbiting at 200 km has a diameter of 35 cm. What is the smallest distance this camera can resolve on the surface of the earth? (Assume a wavelength of 500 nm.)

Answer

Using Rayleigh's criterion and a wavelength of 5.0×10^{-7} m, we find that the distance s that can be resolved is given by $s = r\theta$ where

$$\theta \approx \frac{1.22 \times 5 \times 10^{-7}}{0.35}$$
$$\approx 1.74 \times 10^{-6} \text{ rad}$$
$$\Rightarrow s = r\theta$$
$$= 2 \times 10^{5} \times 1.74 \times 10^{-6} \text{ m}$$
$$= 0.34 \text{ m}$$

Q2

The headlights of a car are 2 m apart. The pupil of the human eye has a diameter of about 2 mm. Suppose that light of wavelength 500 nm is being used. What is the maximum distance at which the two headlights are seen as distinct?

Answer

The resolution of the eye is

$$\theta \approx \frac{1.22 \times 5 \times 10^{-7}}{2 \times 10^{-3}}$$
$$\approx 3 \times 10^{-4} \text{ rad}$$
$$\Rightarrow r = \frac{s}{\theta}$$
$$= \frac{2}{3 \times 10^{-4}}$$
$$= 0.67 \times 10^{4}$$
$$\approx 700 \text{ m}$$

The car should be no more than this distance away.

Q3

The pupil of the human eye has a diameter of about 2 mm and the distance between the pupil and the back of the eye (the retina) where the image is formed is about 20 mm. Suppose the eye uses light of wavelength 500 nm. Use this information to estimate the distance between the receptors in the eye.

Answer

The angular separation, θ, of two objects that can be resolved is, from the answer to Example question 2 above, 3×10^{-4} rad. From Figure 8.5 this is also the angular separation between two receptors on the retina. Thus, the linear separation of the two receptors must be *smaller* than about

$$\begin{aligned} l &= r\theta \\ &= 20 \times 10^{-3} \times 3 \times 10^{-4} \\ &= 6 \times 10^{-6} \text{ m} \end{aligned}$$

As we have seen, the Rayleigh criterion states that two objects are just resolvable as distinct objects if their angular separation is not smaller than the angle θ given by

$$\theta = 1.22\frac{\lambda}{b}$$

In the case of a microscope, the object is placed a distance from the lens (see Figure 8.6) equal to the focal length f of the lens, and so

$$s = f\theta$$

Then the condition for resolution on the object becomes

$$s = 1.22\frac{\lambda f}{b}$$

In practice, $f \approx b$, i.e. these two lengths are of the *same order of magnitude*, and this means that

$$s \approx \lambda$$

(In writing down this formula we neglect the factor of 1.22 because the expression above is only meant to be understood at the level of orders of magnitude.) This states the very important *general result* that:

▶ To resolve a small object of size s, the wavelength λ of light used must be of the same order of magnitude as s or smaller.

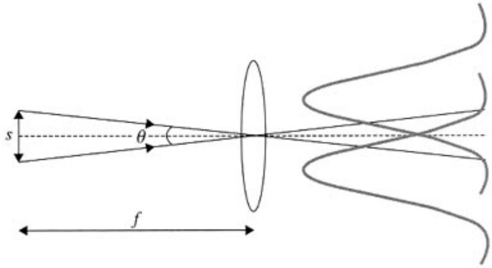

Figure 8.6 The Rayleigh resolution criterion applied to a microscope used to view a very small object.

This illustrates, for example, the operating principle of the electron microscope. To 'see', i.e. resolve, a small object of size, say, 0.01 nm, waves of roughly this wavelength must be used. This means that visible light cannot be used. On the other hand, according to de Broglie, electrons have a wave nature and so they are used in an electron microscope. If the electrons are accelerated to, say, 10^5 V, their kinetic energy will be $E_k = 10^5$ eV $= 1.6 \times 10^{-14}$ J. Using

$$E_k = \frac{p^2}{2m}$$

we find a momentum p of

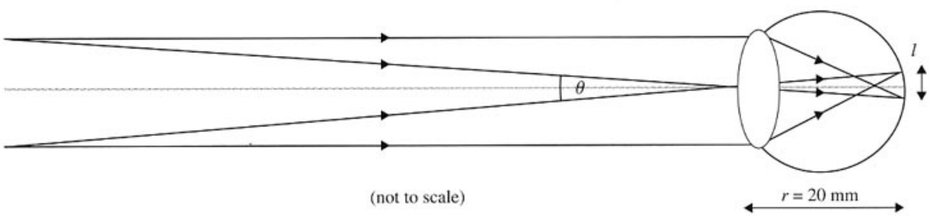

Figure 8.5 The point here is that if the two receptors had a separation larger than 6 μm, the two images would fall on the same receptor and would then appear as one.

$$p = \sqrt{2mE_k}$$
$$= \sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-14}}$$
$$= 1.71 \times 10^{-22}\ \text{N s}$$

and hence a de Broglie wavelength of

$$\lambda = \frac{h}{p}$$
$$= \frac{6.63 \times 10^{-34}}{1.71 \times 10^{-22}}$$
$$= 4 \times 10^{-12}\ \text{m}$$

This is small enough to resolve the size of 0.01 nm. An extension of this general principle of resolution therefore implies that, to resolve the structure of elementary particles, where separations as small as 10^{-18} m are involved, one must use a wavelength of this order of magnitude. If electrons are used, the energy required for the electron is in excess of 1000 GeV. This means that particle physics requires accelerators!

Questions

1 Could a telescope with an objective lens of diameter 20 cm resolve two objects a distance of 10 km away separated by 1 cm? (Assume we are using a wavelength of 600 nm.)

2 The headlights of a car are separated by a distance of 1.4 m. At what distance would these be resolved as two separate sources by a lens of diameter 5 cm if a wavelength of 500 nm is being used? What effect would decreasing the wavelength used have on the distance you just found?

3 Assume that the pupil of the human eye has a diameter of 4.0 mm and receives light of wavelength 5.0×10^{-7} m.

(a) Calculate the smallest angular separation that can be resolved by the eye at this wavelength.

(b) What is the least distance between features on the moon (a distance of 3.8×10^{8} m away) that can be resolved?

4 The Jodrell Bank radio telescope has a diameter of 76 m. Assume that it receives electromagnetic waves of wavelength 21 cm.

(a) Calculate the smallest angular separation that can be resolved by this telescope.

(b) Determine whether this telescope can resolve the two stars of a binary star system that are separated by a distance of 3.6×10^{11} m and are 8.8×10^{16} m from earth (assume a wavelength of 21 cm).

5 The Arecibo radio telescope has a diameter of 300 m. Assume that it receives electromagnetic waves of wavelength 8.0 cm. Determine if this radio telescope will see the Andromeda galaxy (a distance of 2.5×10^{6} light years away) as a point source of light or an extended object. Take the diameter of Andromeda to be 2.2×10^{5} light years.

6 A spacecraft is returning to earth after a long mission far from earth. At what distance from earth will an astronaut in the spacecraft first see the earth and the moon as distinct objects with a naked eye? Take the separation of the earth and the moon to be 3.8×10^{8} m, and assume a pupil diameter of 4.5 mm and light of wavelength 5.5×10^{-7} m.

7 The Hubble Space Telescope has a mirror of diameter 2.4 m.

(a) Estimate the resolution of the telescope assuming that it operates at a wavelength of 5.5×10^{-7} m.

(b) Suggest why the Hubble Space Telescope has an advantage over earth-based telescopes of similar mirror diameter.