Physics IB 12

**Simple Harmonic Motion Problems**

**Starting Points:**

 **Fs = -k∆x** (Hooke’s Law; the force is a “restoring force”)

 **Amplitude** (A): maximum displacement from equilibrium or rest position (m)

 **Period** (T): time to complete one cycle or motion 9s)

 **Frequency** (*f*): # of cycles (motions) per sec (Hz = Hertz)

 F = -kx and F = ma

 -kx = ma

 a = -kA/m (at maximum stretch)

 a = -kx/m (at any distance, x)



The back and forth motion of the mass is called Simple Harmonic Motion (SHM). A graph of position of the mass vs time is sinusoidal in nature and is shown below. A graph of velocity of the mass vs time is also sinusoidal in nature as shown below:



Examine the graph and the image of the mass shown earlier and convince yourself how the 2 are related.

**Kinematics of Simple Harmonic Motion**

The following graph shows the position vs time graph for SHM:

Two equations that can describes this motion are:

 Position is a cosine function:

 x(t) = A cos (2πt/T)

 Velocity is a Sine function:

 *v*x = -*v*max sin (2πt/T)



From previously: 2πf = 2π/T = ω (angular velocity) and:

 x(t) = A cos (ωt) & *v*x = -*v*max sin (ωt)

 And that:

 vmax = 2π/T = 2π*f =* ωA (which occurs when position x = A)

**Example 1**

An air-track glider is attached to a spring pulled 20 cm to the right and released at t = 0. It makes 15 oscillations in 10 s.

1. What is the period and frequency of oscillation?

*T = 10 s /15 = 0.667 s & f = 15/10 s = 1.5 Hz*

1. What is the objects maximum speed?

*vmax = 2πA/T = 2π(0.20 m)/0.667 s = 1.88 m/s*

1. What are the position and velocity at t = 0.80 s? *(Note: Must use radians in cos and sine calc)*

*Position: x(t) = A cos (2πt/T); x = 0.20 m(cos (2π0.80 s)/0.667 s) = 0.062 m*

*Velocity: vx = -vmax sin (2πt/T); vx = -1.88 m/s sin (2π0.80 s/0.667 s) = - 1.79 m/s*

**Example 2**

A mass oscillating in simple harmonic motion starts at x = A and has period T. At what time, as a fraction of T, does the object first pass through x = ½ A?

 *x(t) = A cos (2πt/T); x = (A/2) = A cos (2πt/T);*

*now solve for the time at which this position is reached:*



*t = (T/2π) cos-1 (1/2) = (T/2π)( π/3) = (1/6) T*

**Simple Harmonic Motion and Circular Motion**



**A**

**θ**

**A**

**-A**

**θ**

 **x = A**

A ball at the end of a string travelling in a vertical circle will project a shadow onto the x-axis as shown. The shadow will only move back and forth exhibiting SHM.

The displacement on the x-axis as the ball moves in this circle is given by: x = A cos θ.

Starting Point: x = A cos θ but remember ω = θ/t and θ = ωt

 x = A cos ωt and ω = 2π/T

 x = A cos (2π/T) t or x = A cos (2πt/T)

 Which is the same as the equation for position for simple harmonic motion!

**Energy in Simple Harmonic Motion**

Potential Energy in a spring: **U = ½ kx2**

 Kinetic Energy: **Ek = ½ mv2**

 **Total Energy: ET = Ek  + U**



At some displacement, x = ½ xm;

ET = U + Ek = ½ kxm2 + ½ m*v*x2

At maximum displacement, x = -A;

ET = U = ½ kx2 (Ek = 0)

At minimum displacement, x = 0;

ET = Ek = ½ mvmax2  (U = 0)

Since the system’s mechanical energy is conserved (no energy is lost due to friction):

 ET = Ek + U = ½ m*v*x2 + ½ kx2

At maximum displacement (x = + A) all the energy is all potential:

 U (at x = + A) = ½ kx2 = ½ kA2

At the mid-point (x = 0 & *v*x = + *v*max) all the energy is kinetic:

 Ek (x = 0) = ½ m*v*max2

Since energy is conserved the energy at maximum and minimum must be the same, and hence:

 Ek = U

 1/2mvmax2 = 1/2kx2 = 1/2kA2 (and x = A at maximum displacement)

 vmax2 = (k/m) A2

 vmax  = √(k/m) A

 Remember from SHM: vmax = 2πA/T = 2πA*f* = ωA

 Therefore: √(k/m) A = ωA

 And ω = √(k/m)

 Since T = 2 πω and *f* = (1/2 πω)

 T = 2 π √(m/k) & *f* = (1/2 π) √(k/m)

Because energy is conserved we can write:

ET = 1/2m*v*x2 + 1/2kx2  = 1/2kA2 = 1/2m*v*max2

Any pair may be useful, depending on the known information:

For example, you can use the amplitude A, to find the speed at any point x by combining the first and second expression:

v = √K/m(A2 – x2) = ω √(A2 – x2)

Or you can use the first and second expression to find the amplitude from the initial conditions xo and *v*o:

 A = √(xo2 + m*v*o2/k) = √(xo2 + *v*o2/ ω); (ω = √k/m )

**Example 3**

A 500 g block on a spring is pulled a distance of 20 cm and released. The subsequent oscillations are measured to have a period of 0.80 s. At what position or positions is the block’s speed 1.0 m/s?

Solution: Energy is conserved!

 1/2m*v*x2 + 1/2kx2  = 1/2kA2

Solving for x:

 ½ m*v*x2 + ½ kx2  = ½ kA2

 m*v*x2 + kx2  = kA2

 kx2  = kA2 - m*v*x2

 x2  = (kA2/k) – (m*v*x2/k)

 x2  = (A2) – (m*v*x2/k)

 x = √(A2 – m*v*x2/k) = √(A2 – (*v*x/ ω) 2) (where k/m = ω2)

 and ω = 2π/T = 2π/0.80 s = 7.85 rad/s

 x = √(0.202 – (1.0/7.85)2) = + 0.154 m

**Stop and Think Questions:**

This is the position vs time graph of a mass on a spring. The mass is pulled back a distance of 10 m and then let go.



1. What is the position of the mass at the dotted line?
2. Is the velocity of the mass a maximum or minimum at this position?
3. What direction is the force acting on the spring at this point?
4. State a time that the mass would have a maximum speed?
5. State a time where the mass would have a maximum potential energy?
6. State a time where the mass would have a minimum potential energy?
7. State a time where the mass would have a maximum kinetic energy?
8. State a time where the mass would have a minimum kinetic energy?
9. Calculate the period and frequency of the mass?
10. Use ω = 2πf = √(k/m) to determine the spring constant for this mass of 3.0 kg.

**Vertical Oscillations**

A spring is hung vertically from a ceiling. When a mass is added to that spring and pulled down and released simple harmonic motion results.



Since the mass is not moving (is at equilibrium) at (ii):

 Fg = Fs

 mg = k∆L

 ∆L = mg/k

Fs

∆L

Pull the mass down further (iii) and then let the mass move; it will oscillate up and down in simple harmonic motion. The displacement, y, of the mass is described by the exact same equation shown before:

 y (t) = A cos ωt;

 where ω = 2πf = √(k/m)

Fg

**Example 4**

A 200 g block hangs stationary from a spring pulling the spring downwards a distance of 19.6 cm. The block is then pulled down a distance of 30 cm and released. Where is the block and what is its velocity 3.0 s later?

Solution:

Find k: k = mg/∆L = 0.200 kg x 9.8/0.196 m = 10 N/m

 y (t) = A cos ωt = A cos (√(k/m))t = 10.4 cm cos (√(10/0.200)) x 3.0

 *y* = -7.4 cm (above the equilibrium position) (ω = (√(k/m = 7.07 rads)

 *v*x = ωA sin(ωt) = 7.07 x 10.4 x sin (7.07 x 3.0) = 52 cm/s (ω = (√(k/m = 7.07 rads)

**The Pendulum**

A mass hanging from a string attached to a ceiling makes a very good oscillator.

Free-Body Diagram for the mass, m:

FT

Ø

L

Wt

m

s

Arc length

Wt = mg sin Ø

(component of gravity force)

Ø = s/L

Fg

Therefore: net force = Fnet = mg sin Ø and since ma = mg sin Ø & a = g sin Ø

For small angles of Ø there is an approximation that works really well:

 When using radians: s = r sin Ø ≈ r Ø (sin Ø ≈ Ø) (good for up to 0.10 rad (10o)

s = r Ø

r

Ø

h = r sin Ø

*l* = r cos Ø

 Sin Ø ≈ Ø = s/L (r = L)

 The angular frequency, ω = 2πf = √g/L

And therefore frequency and period for a pendulum would be:

 f = 1/2π √g/L and T = 2π√L/g

**Example 5**

What length and mass of a pendulum has a period of exactly 1 s?

Solution: Mass can be any value!

 T = 2π√L/g therefor: L = g (T/2 π)2 = 9.80 (1.0/2 x 3.14)2 = 0.248 m